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Exact eigenstates of the two-photon Jaynes-Cummings model with the counter-rotating term

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Abstract. We have investigated the eigenenergy spectrum of the two-photon Jaynes-Cummings (JC) model with and without the rotating-wave approximation (RWA). Our analysis has indicated that the counter-rotating term dramatically changes the nature of the RWA energy spectrum and that the non-RWA spectrum can be approximated by the RWA spectrum only in the range of a sufficiently small coupling constant. Furthermore, unlike the one-photon counterpart, the two-photon JC model without the RWA is well defined only if the coupling parameter is below a certain critical value. As a result, the dynamics of the two-photon JC model without the RWA is significantly different from its RWA counterpart. For instance, the counter-rotating term can dramatically enhance the field squeezing effect. Besides, we would expect that the quantum dynamics of the two-photon JC model without the RWA is qualitatively different from that of the usual one-photon case.

PACS. 42.50.Vk Mechanical effects of light on atoms, molecules, electrons, and ions – 32.90.+a Other topics in atomic properties and interactions of atoms and ions with photons (restricted to new topics in section 32)

1 Introduction

The interaction between radiation and matter is a central problem in quantum optics. The simplest physical situation can be successfully described by a rather simplified but non-trivial model proposed by Jaynes and Cummings more than three decades ago [1], which idealizes the real situation by concentrating on the near-resonance linear coupling between a single two-level atomic system and a quantized radiation mode ($\hbar = 1$):

$$H = \omega_0 S_z + \omega a^\dagger a + \epsilon \left(a^\dagger S_- + a S_+ \right) , \qquad (1)$$

where the radiation mode of frequency ω is described by the bosonic operators a and $a^\dagger$, the two atomic levels separated by an energy difference ω_0 are represented by the spin-half operators S_z and $S_\pm$, and the atom-field coupling strength is measured by the positive parameter ϵ. Despite its simplicity, the Jaynes-Cummings (JC) model is of great significance because recent technological advances have enabled us to experimentally realize this rather idealized model [2–5] and to verify some of the theoretical predictions. Stimulated by the success of the JC model, more and more people have paid special attention to extending and generalizing the model in order to explore new quantum effects [6]. One possible generalization is the two-photon JC model:

$$H = \omega_0 S_z + \omega a^\dagger a + \epsilon \left(a^{\dagger 2} S_- + a^2 S_+ \right) . \qquad (2)$$

Such a generalization is of considerable interest because of its relevance to the study of the coupling between a single atom and the radiation field with the atom making two-photon transitions [7–10]. It is noted that the quantum dynamics of the two-photon JC model is qualitatively different from that of the usual single-photon JC model. Recent advances in the two-photon micromaser have also made such investigations not for purely theoretical interests only, and have stimulated more and more attention to this subject [11–13].

As in the original JC model, the two-photon JC model is analytically solvable due to the neglect of the so-called *counter-rotating terms* : $\epsilon \left(a^{\dagger 2} S_+ + a^2 S_- \right)$. Strict analysis of the validity of this *rotating-wave approximation* (RWA) is, however, not usually considered in concrete applications, and the range of the system parameters where the results are meaningful remains uncertain. Therefore, it is of great interest to analyse the exact solutions of the more complete models that contain the counter-rotating terms for a wide range of the system parameters, and compare them with the RWA results. Such studies are useful for determining the limits of validity of the RWA. For instance, in a recent study of the one-photon JC model beyond the RWA, it has been shown that the eigenenergy spectrum of the system can be approximated by the RWA results only in the range of a sufficiently small coupling constant, and that the width of this range decreases as one goes to the highly excited states [14]. Furthermore, Ford and O'Connell recently pointed out that although the RWA is

an integral part of the foundations of quantum optics, yet it can have a very serious defect, *viz.* the RWA spectrum has no lower bound for all models of physical interest [15]. In other words, the RWA model is ill-defined. Besides, in our recent study of the k-photon JC model without the RWA, we also observed that the model is in fact undefined for $k > 2$ and qualitatively different from the RWA counterpart [16]. In this paper we would like to investigate the effects of the counter-rotating term in the two-photon JC model as well. In order to solve the two-photon JC model in which the counter-rotating term is included, we need to resort to the numerical diagonalization method because analytical solutions are not available. Nevertheless, the presence of some unitary transformation which decouples the spin degree of freedom from the bosonic one and the $SU(1,1)$ Lie symmetry of the system helps bring the Hamiltonian into block-diagonal form, and thus the numerical calculations are considerably simplified, as shown in the following section. The scheme of this paper is as follows. In section 2 we shall describe how to obtain the eigenenergy spectrum of the two-photon JC Hamiltonian with and without the counter-rotating term. Finally, in the last section the numerical results are discussed and a brief conclusion is presented.

2 Two-photon JC model with and without RWA

To begin with, let us introduce the operators K_+, K_- and K_0:

$$K_+ = \frac{1}{2}a^{\dagger 2},\ K_- = K_+^\dagger = \frac{1}{2}a^2,\ K_0 = \frac{1}{4}\left(2a^\dagger a + 1\right). \quad (3)$$

These three operators form a closed Lie algebra $SU(1,1)$, which is defined by the commutation relations [17]

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0\,. \quad (4)$$

The corresponding Casimir operator C is given by

$$C = K_0^2 - \frac{1}{2}\left(K_+K_- + K_-K_+\right) = -\frac{3}{16}\,, \quad (5)$$

which has the eigenvalue $k\,(k-1)$ for a unitary irreducible representation (UIR). The parameter k is the so-called Bargmann index. For the UIR known as the positive discrete series $\mathcal{D}^+(k)$, the states $|m,k\rangle$ diagonalize the compact operator K_0:

$$K_0|m,k\rangle = (m+k)\,|m,k\rangle\,, \quad (6)$$

for $k > 0$ and $m = 0, 1, 2, \ldots$ [Note that in the single-mode bosonic realization of the $SU(1,1)$ Lie algebra k can be equal to $1/4$ or $3/4$. For $k = 1/4$ we obtain the even-parity states of the bosonic mode, whereas $k = 3/4$ gives the odd-parity states. The vacuum state for the bosonic mode is apparently the state $|0, 1/4\rangle$.] The operators K_+ and K_- are Hermitian conjugates of each other and act as raising and lowering operators respectively within $\mathcal{D}^+(k)$, *i.e.*

$$\begin{aligned} K_+|m,k\rangle &= \sqrt{(m+1)\,(m+2k)}\;|m+1,k\rangle\,,\\ K_-|m,k\rangle &= \sqrt{m\,(m+2k-1)}\;|m-1,k\rangle\,. \end{aligned} \quad (7)$$

The corresponding $SU(1,1)$ generalized coherent states $|\alpha;k\rangle$ are defined as

$$\begin{aligned} |\alpha;k\rangle &= \exp\left(\alpha K_+ - \alpha^* K_-\right)|0,k\rangle = \\ &\exp\left[\frac{1}{2}\left(\alpha a^{\dagger 2} - \alpha^* a^2\right)\right]|0,k\rangle\,. \end{aligned} \quad (8)$$

For $k = 1/4$, the coherent state $|\alpha; k = 1/4\rangle$ is simply the well-known single-mode squeezed vacuum state with the squeezing parameter α.

In terms of the $SU(1,1)$ generators, we may re-write the Hamiltonian in equation (2) as

$$H = H_0 + V\,, \quad (9)$$

where

$$\begin{aligned} H_0 &= \omega_0 S_z + 2\omega\left(K_0 - \frac{1}{4}\right),\\ V &= 2\epsilon\left(K_+S_- + K_-S_+\right). \end{aligned} \quad (10)$$

Recognizing that the operator of the total number of excitations

$$\mathcal{N} = \left(K_0 - \frac{1}{4}\right) + S_z + \frac{1}{2}\,, \quad (11)$$

whose eigenvalues are non-negative definite, commutes with all operators conserving the total number of excitations, *e.g.* $[\mathcal{N}, H_0] = [\mathcal{N}, V] = 0$, we can easily show that the Hamiltonian H can be diagonalized by the dressing unitary operator

$$T = \exp\left\{\gamma\left(K_+S_- - K_-S_+\right)\right\}, \quad (12)$$

where

$$\begin{aligned} \beta &= \sqrt{\left(\mathcal{N} - \frac{3}{4}\right)\left(\mathcal{N} + \frac{1}{4}\right) - C} = \sqrt{\mathcal{N}\left(\mathcal{N} - \frac{1}{2}\right)},\\ \tan(\theta) &= -\frac{2\epsilon\theta}{\gamma\,(2\omega - \omega_0)} = -\frac{4\epsilon\beta}{2\omega - \omega_0}\,. \end{aligned} \quad (13)$$

The transformed Hamiltonian $\widetilde{H} \equiv T^\dagger H T$ takes the diagonal form

$$\widetilde{H} = \left(2\omega + \frac{\Delta}{\gamma}\right)S_z + 2\omega\left(K_0 - \frac{1}{4}\right), \quad (14)$$

where $\Delta = \sqrt{(2\epsilon\theta)^2 + \gamma^2\,(2\omega - \omega_0)^2}$. The eigenstates of $\widetilde{H}$ are simply given by $|m,k,\sigma\rangle \equiv |m,k\rangle|\sigma\rangle$ for $\sigma = \uparrow$ or $\downarrow$,

$k = 1/4$ or $3/4$, and $m = 0, 1, 2, \ldots$, and the corresponding eigenenergies can be straightforwardly found in the form

$$\begin{aligned}
E_{m,\downarrow}^{k=1/4} &= 2\omega\left(m-\frac{1}{2}\right) \\
&\quad -\frac{1}{2}\sqrt{16\epsilon^2\left(m-\frac{1}{2}\right)m+\left(2\omega-\omega_0\right)^2}\,, \\
E_{m,\downarrow}^{k=3/4} &= 2\omega m-\frac{1}{2}\sqrt{16\epsilon^2 m\left(m+\frac{1}{2}\right)+\left(2\omega-\omega_0\right)^2}\,, \\
E_{m,\uparrow}^{k=1/4} &= 2\omega\left(m+\frac{1}{2}\right) \\
&\quad +\frac{1}{2}\sqrt{16\epsilon^2\left(m+\frac{1}{2}\right)(m+1)+\left(2\omega-\omega_0\right)^2}\,, \\
E_{m,\uparrow}^{k=3/4} &= 2\omega\left(m+1\right) \\
&\quad +\frac{1}{2}\sqrt{16\epsilon^2\left(m+1\right)\left(m+\frac{3}{2}\right)+\left(2\omega-\omega_0\right)^2}\,. \quad (15)
\end{aligned}$$

The spin-dependent part in equation (14) can be interpreted as the dressed atom Hamiltonian, and $\widetilde{\omega}_0 \equiv 2\omega + \Delta/\gamma$ represents the renormalized energy difference between the two atomic levels in comparison with H_0 in equation (10). The procedure developed so far is exact and can be generalized to the multimode case without difficulty [18].

Now we consider the effect of the counter-rotating term $\epsilon(a^{\dagger 2}S_+ + a^2 S_-)$ on the two-photon JC model. Incorporating the counter-rotating term into the two-photon JC Hamiltonian, we obtain

$$\mathcal{H} = \omega_0 S_z + 2\omega\left(K_0-\frac{1}{4}\right) + 4\epsilon\left(K_+ + K_-\right)S_x\,. \quad (16)$$

So far as we know, this Hamiltonian cannot be diagonalized analytically; and thus one must resort to the numerical approach to calculate its eigenstates and eigenenergies. To facilitate the numerical diagonalization, we introduce here the unitary transformation

$$R = \exp\left\{-i\pi\left(S_x-\frac{1}{2}\right)\left(K_0-\frac{1}{4}\right)\right\}, \quad (17)$$

which enables us to decouple the spin degree of freedom from the boson mode. The unitary operator R transforms the annihilation operator K_- and the spin operator S_z as follows:

$$\begin{aligned}
R^\dagger K_- R &= 2K_- S_x\,, \\
R^\dagger S_z R &= \cos\left[\pi\left(K_0-\frac{1}{4}\right)\right]S_z + \sin\left[\pi\left(K_0-\frac{1}{4}\right)\right]S_y\,. \quad (18)
\end{aligned}$$

Using these relations, we apply the unitary transformation to the Hamiltonian $\mathcal{H}$ and obtain

$$\begin{aligned}
\widetilde{\mathcal{H}} \equiv R^\dagger \mathcal{H} R &= \omega_0 \cos\left[\pi\left(K_0-\frac{1}{4}\right)\right]S_z \\
&\quad +\omega_0 \sin\left[\pi\left(K_0-\frac{1}{4}\right)\right]S_y \\
&\quad +2\omega\left(K_0-\frac{1}{4}\right)+2\epsilon\left(K_+ + K_-\right)\,. \quad (19)
\end{aligned}$$

It is not difficult to see that within the subspace of the Bargmann index $k = 1/4$ the transformed Hamiltonian is reduced to

$$\begin{aligned}
\widetilde{\mathcal{H}}_{1/4} &= \omega_0 \cos\left[\pi\left(K_0-\frac{1}{4}\right)\right]S_z \\
&\quad +2\omega\left(K_0-\frac{1}{4}\right)+2\epsilon\left(K_+ + K_-\right)\,, \quad (20)
\end{aligned}$$

whereas for $k = 3/4$ we have

$$\begin{aligned}
\widetilde{\mathcal{H}}_{3/4} &= \omega_0 \sin\left[\pi\left(K_0-\frac{1}{4}\right)\right]S_y \\
&\quad +2\omega\left(K_0-\frac{1}{4}\right)+2\epsilon\left(K_+ + K_-\right)\,. \quad (21)
\end{aligned}$$

Obviously, in both cases the spin degree of freedom and the boson mode are decoupled. Eigenstates of each of these two sectors are simply given by $|M\rangle|\phi_l\rangle$, where $|M\rangle$ is an eigenstate of the spin operator and $|\phi_l\rangle$ the l-th eigenstate of the one-body bosonic Hamiltonian h:

$$h = M\omega_0\left(-1\right)^{K_0-k} + 2\omega\left(K_0-\frac{1}{4}\right) + 2\epsilon\left(K_+ + K_-\right)\,, \quad (22)$$

for $M = \pm 1/2$. As a result, the Hilbert space of $\widetilde{\mathcal{H}}$ can be divided into four independent subspaces, each of which is characterized by the Bargmann index k and the corresponding spin quantum number M. With respect to the basis states $|m, k\rangle$ in each subspace, the matrix elements of h are given by

$$\begin{aligned}
\langle n,k|h|m,k\rangle &= \left[M\omega_0\left(-1\right)^m + 2\omega\left(m+k-\frac{1}{4}\right)\right]\delta_{n,m} \\
&\quad +2\epsilon\left[\sqrt{(m+1)(m+2k)}\delta_{n,m+1}\right. \\
&\quad \left.+\sqrt{m(m+2k-1)}\delta_{n,m-1}\right]\,. \quad (23)
\end{aligned}$$

This matrix can be easily diagonalized by standard numerical methods in each subspace to yield the eigenenergy spectrum of the system.

3 Numerical results and discussion

Before we go into the details of the numerical results, let us first examine some of the basic properties of the system.

First of all, there exists a conserved quantity $\mathit{\Pi}$ associated with the Hamiltonian $\mathcal{H}$:

$$\mathit{\Pi} = \exp\left\{i\pi\left[\left(K_0 - \frac{1}{4}\right) + S_z + \frac{1}{2}\right]\right\} = -2S_z \exp\left[i\pi\left(K_0 - \frac{1}{4}\right)\right], \quad (24)$$

i.e. $[\mathcal{H}, \mathit{\Pi}] = 0$. Applying the unitary transformation R to $\mathit{\Pi}$, we obtain

$$\widetilde{\mathit{\Pi}} \equiv R^\dagger \mathit{\Pi} R = -2\left\{\cos\left[\pi\left(K_0 - \frac{1}{4}\right)\right] S_z + \sin\left[\pi\left(K_0 - \frac{1}{4}\right)\right] S_y\right\} \times \exp\left[i\pi\left(K_0 - \frac{1}{4}\right)\right] \quad (25)$$

Within the subspace of the Bargmann index $k = 1/4$ the transformed operator $\widetilde{\mathit{\Pi}}$ is reduced to

$$\widetilde{\mathit{\Pi}}_{1/4} = -2S_z\,, \quad (26)$$

whereas for $k = 3/4$ we have

$$\widetilde{\mathit{\Pi}}_{3/4} = -i2S_y\,. \quad (27)$$

Hence, the fact that the eigenstates of the transformed Hamiltonian in the two Bargmann sectors are eigenstates of the spin operators originates from the basic compatibility of $\mathcal{H}$ and $\mathit{\Pi}$.

Next, for $\omega_0 = 0$ it is not difficult to show that the one-body Hamiltonian h in equation (22) can be diagonalized by the unitary $SU(1,1)$ displacement transformation $S = \exp\left[-(1/2)\tanh^{-1}(2\epsilon/\omega)\cdot(K_+ - K_-)\right]$:

$$\tilde{h} = S^\dagger h S = 2\widetilde{\omega}K_0 - \frac{\omega}{2}\,, \quad (28)$$

where $\widetilde{\omega} = \omega\sqrt{1-(2\epsilon/\omega)^2}$. One should notice that the function $\tanh^{-1}(2\epsilon/\omega)$ is well defined only if $2\epsilon/\omega < 1$; in other words, for $2\epsilon/\omega > 1$, there does not exist any unitary transformation S which can diagonalize the Hamiltonian h (with $\omega_0 = 0$). This can be easily understood as follows. In terms of the bosonic creation and annihilation operators, the Hamiltonian h (with $\omega_0 = 0$) can be expressed as

$$h = \omega a^\dagger a + \epsilon\left(a^{\dagger 2} + a^2\right)\,. \quad (29)$$

Defining the "position" and "momentum" operators of the boson mode as

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \quad p = \frac{1}{i}\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right)\,, \quad (30)$$

respectively, we may re-write h in equation (29) as

$$h = \frac{p^2}{2\widetilde{m}} + \frac{1}{2}\widetilde{m}\widetilde{\omega}^2 x^2 - \frac{\omega}{2}\,, \quad (31)$$

where $\widetilde{m} = [1-(2\epsilon/\omega)]^{-1}$. Clearly, for $2\epsilon/\omega < 1$, the Hamiltonian h corresponds to a quantum harmonic oscillator of mass $\widetilde{m}$ and frequency $\widetilde{\omega}$, and thus can be diagonalized using the basis states of $\hat{n} \equiv a^\dagger a$. On the contrary, for $2\epsilon/\omega > 1$, the Hamiltonian h represents an inverted oscillator in the momentum space, and thus cannot be diagonalized using the basis states of $\hat{n}$ because its eigenstates are not normalizable. In other words, the basis states of $\hat{n}$ are not *analytic vectors* of h, and the operator h is not *self-adjoint* in the space of Gaussian measure [19,20]. The above analysis still holds for $\omega_0 \neq 0$ because the first term in equation (22) is a bounded operator and the basis states

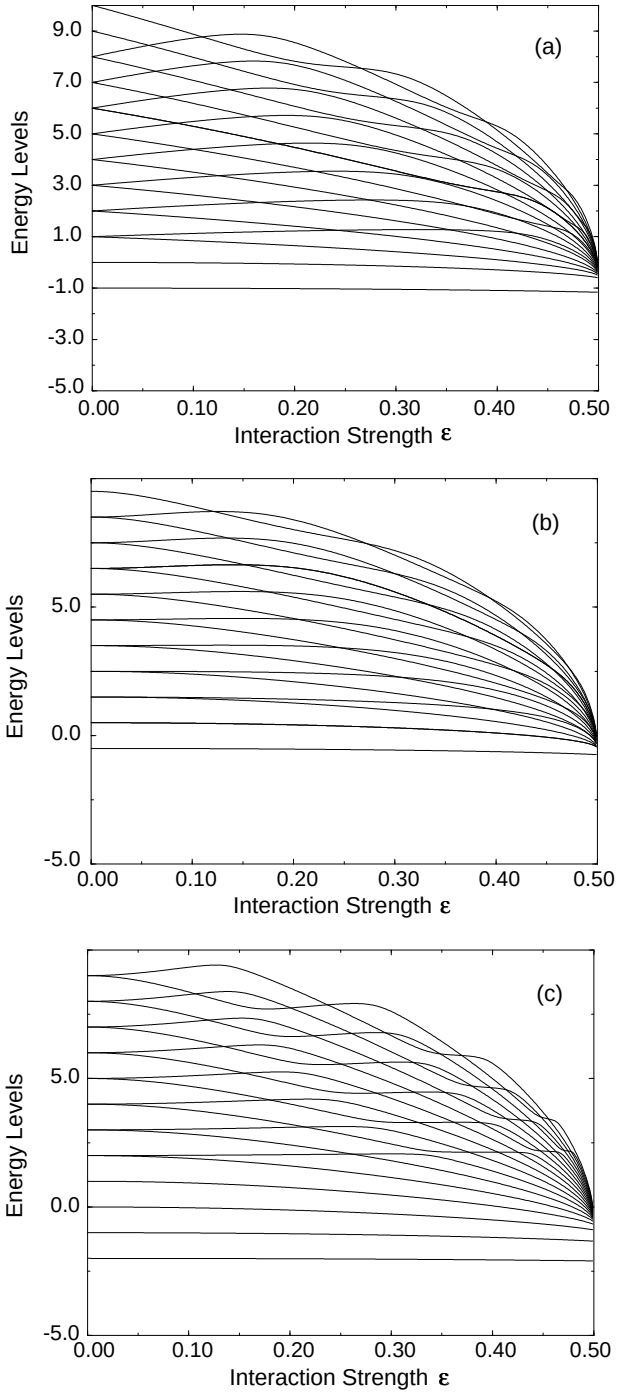

Fig. 1. Energy spectrum of the two-photon JC model without RWA for different interaction strength ϵ. Only the lowest five eigenstates in each subspace are shown. (a) $\omega = 1$, $\omega_0 = 2$; (b) $\omega = 1$, $\omega_0 = 1$; (c) $\omega = 1$, $\omega_0 = 4$.

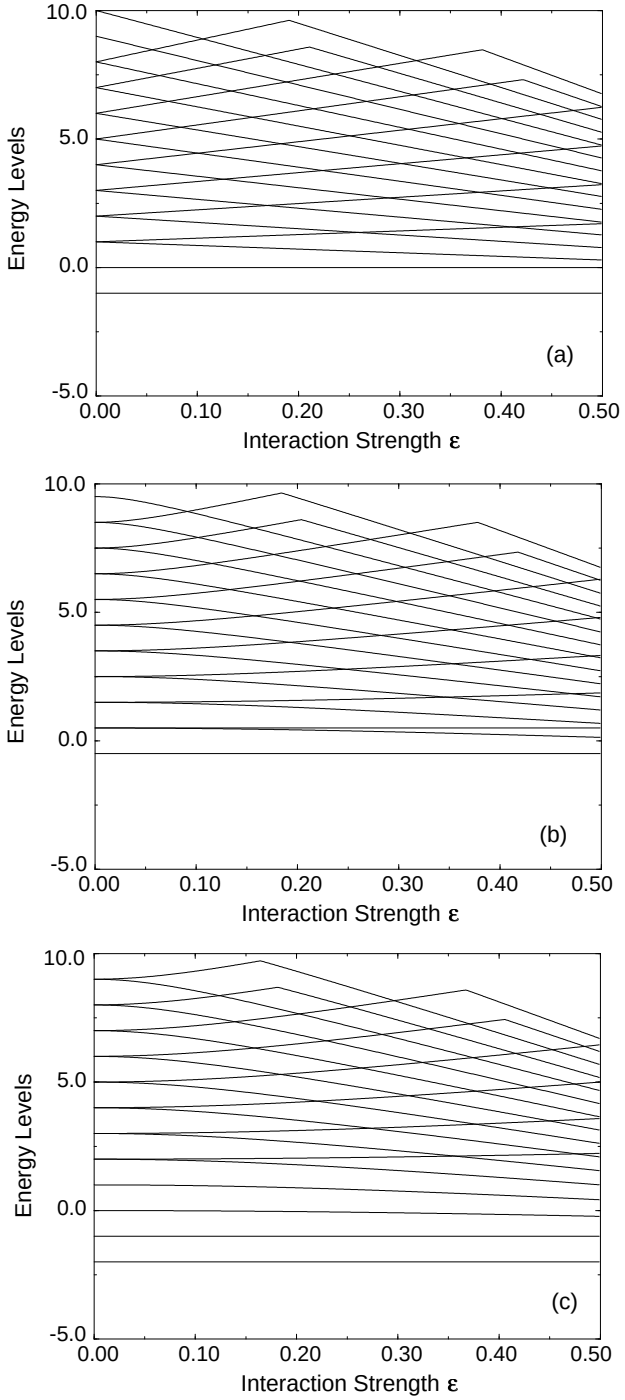

Fig. 2. Energy spectrum of the two-photon JC model with RWA for different interaction strength ϵ. Only the lowest five eigenstates in each subspace are shown. (a) $\omega = 1$, $\omega_0 = 2$; (b) $\omega = 1$, $\omega_0 = 1$; (c) $\omega = 1$, $\omega_0 = 4$.

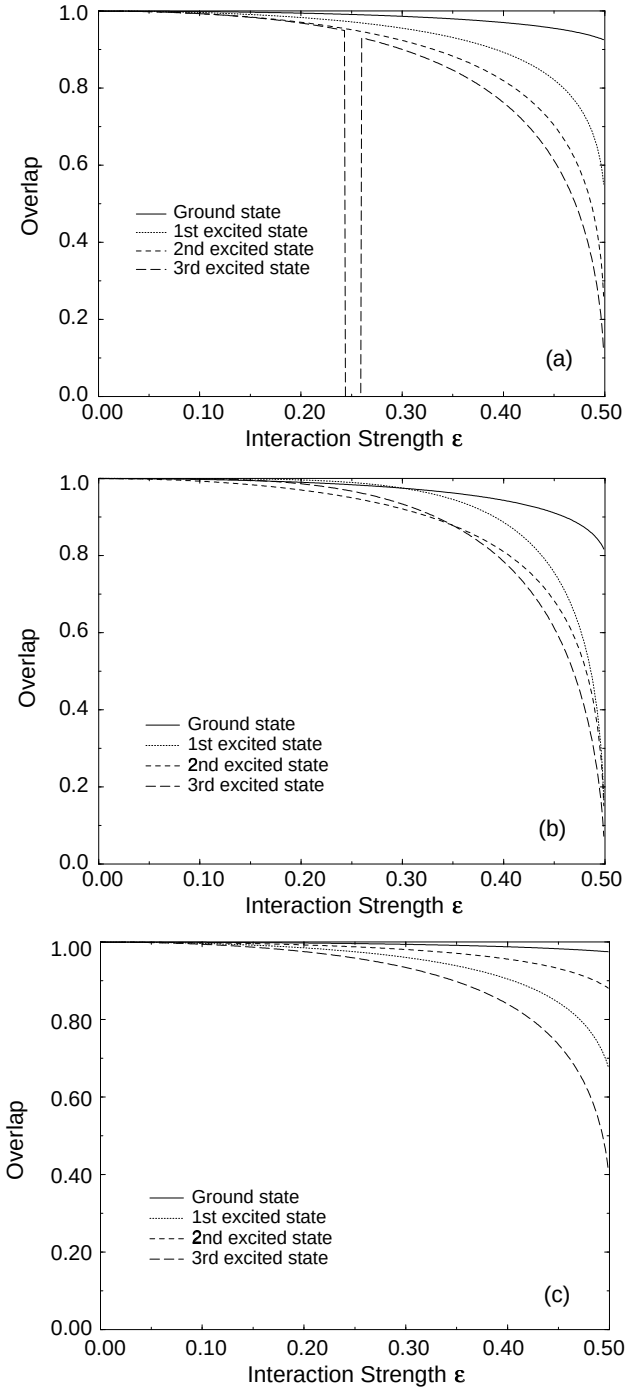

Fig. 3. Overlap (square of the inner product) between a RWA eigenstate and its non-RWA counterpart. (a) $\omega = 1$, $\omega_0 = 2$; (b) $\omega = 1$, $\omega_0 = 1$; (c) $\omega = 1$, $\omega_0 = 4$.

of $\hat{n}$ are apparently analytic vectors of this bounded operator. Accordingly, the two-photon JC model without the RWA is well defined only if $2\epsilon/\omega < 1$, and this is in sharp contrast to the RWA result.

In Figure 1 we show the exact eigenenergy spectrum of the system without the RWA as a function of the coupling parameter ϵ for different values of ω_0. For convenience, we have set the energy unit such that $\omega = 1$. We also plot the RWA results in Figure 2 for comparison. It is clear that both the RWA and the non-RWA spectra exhibit complicated patterns of energy-level crossing as the coupling parameter ϵ varies, and that the eigenenergy spectrum of the system can be approximated by the RWA spectrum only in the range of a sufficiently small coupling parameter. The range of validity of the RWA can be determined in a more quantitative manner by simply examining the overlap of the exact eigenstate and its RWA

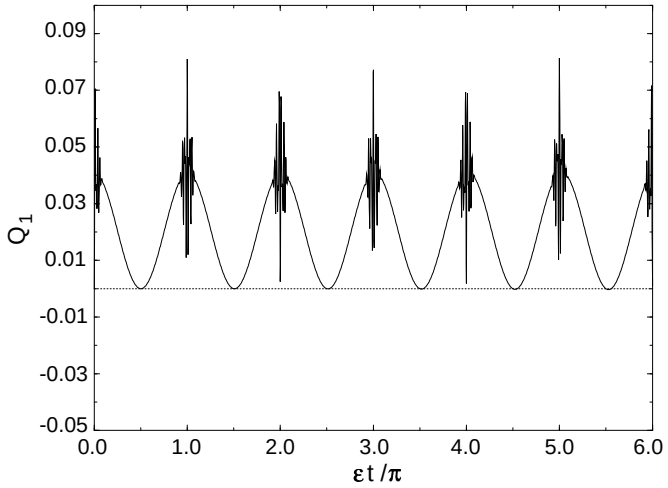

Fig. 4. The time evolution of $Q_1(t)$ within the RWA and at resonance.

counterpart as shown in Figure 3. Of particular interest is the result displayed in Figure 3(a). It illustrates that for a given value of the coupling parameter ϵ, an exact eigenstate in the non-RWA spectrum and its RWA counterpart can belong to completely different subspaces so that the two states do not overlap. Fundamentally this phenomenon is associated with the disparity between the pattern of the energy-level crossing in the non-RWA spectrum and that in the RWA spectrum. In fact, such orthogonality of corresponding eigenstates is found to be prevalent among the highly excited states. Thus, for dynamical processes which involve the collective evolution of the eigenstates, the description based upon the RWA may deviate markedly from the actual behaviour of the system, especially when the coupling parameter is not small. As the coupling parameter increases, the energy level spacing decreases monotonically. In particular, as $2\epsilon/\omega \to 1^-$, the higher energy levels dramatically collapse towards the lower ones. This can be attributed to the fact that the eigenenergy spectrum of $\tilde{h}$ in equation (28) approaches a continuum as $\widetilde{\omega} \to 0^+$, and that the spin-dependent term in equation (22) can help separate some of the lower energy levels only. The splitting of the lower energy levels is observed to be more effective for large ω_0. It is also interesting to point out that in the case of $\omega_0 = 1$ the first and second excited states of the system without the RWA are degenerate for all the allowed values of ϵ, and that the RWA removes this accidental degeneracy. Furthermore, beyond the critical value of ϵ, *i.e.* $\epsilon = 1/2$, it is observed that the numerical diagonalization using the basis states of $\hat{n}$ does not give any converged result at all. This is consistent with our analysis above, and thus confirms that the two-photon JC model without the RWA is well defined only if $2\epsilon/\omega < 1$. Accordingly, the two-photon JC model without the RWA is qualitatively different from the one-photon counterpart which is valid for all values of the coupling parameter [14].

Finally, we shall apply the above results to study the effects of the counter-rotating term on the dynamics of the system, in particular the time evolution of the field squeezing. We define the two amplitude operators a_1 and a_2 of the quadrature of the field as follows:

$$a_1 = \frac{1}{2}\left[a\exp\left(i\omega t\right) + a^\dagger \exp\left(-i\omega t\right)\right] ,$$
$$a_2 = \frac{1}{2i}\left[a\exp\left(i\omega t\right) - a^\dagger \exp\left(-i\omega t\right)\right] , \qquad (32)$$

with $[a_1, a_2] = i/2$. Field squeezing occurs whenever

$$Q_k(t) = 4\left[\langle(\Delta a_k)^2\rangle_t - \frac{1}{4}\right] < 0, \quad k = 1, 2 , \qquad (33)$$

where $\langle(\Delta a_k)^2\rangle_t = \langle a_k^2\rangle_t - \langle a_k\rangle_t^2$. The initial state $|\psi(0)\rangle$ of the system is taken to be the product state $|\psi_{\rm field}\rangle \otimes |\psi_{\rm atom}\rangle$, where $|\psi_{\rm field}\rangle$ is just a coherent state $|\alpha\rangle$ of the field and $|\psi_{\rm atom}\rangle$ is a symmetric linear combination of the spin states $|\pm 1/2\rangle$ of S_z. In Figure 4 we show the time evolution of the parameter $Q_1(t)$ for $\alpha = \sqrt{50}$ within the RWA and at resonance. No field squeezing is observed. In Figure 5 the non-RWA time evolution of $Q_1(t)$ is plotted for a small value of coupling parameter ϵ, *i.e.* 0.005, at resonance. As expected, in the presence of the counter-rotating term, we observe fast oscillations in the evolution of $Q_1(t)$. Furthermore, it is found that the counter-rotating term can cause significant improvement on the field squeezing. The enhancement not only increases with α, but it also grows rapidly with time. So the two-photon JC model without the RWA may serve as a good candidate for preparing quantum states with high degree of field squeezing. In order to obtain a deeper understanding of the effects of the counter-rotating term on the field squeezing, we plot the parameter $Q_1(t)$ for different values of ϵ in Figure 6 as well. We find that the increase in the coupling strength can enhance the field squeezing. However, there exists a critical value of the coupling strength ϵ_c, beyond which the field squeezing starts diminishing. These results are completely different from those with the RWA.

In conclusion, we have investigated the eigenenergy spectrum of the two-photon JC model with and without the RWA. Our analysis has indicated that the counter-rotating term dramatically changes the nature of the RWA energy spectrum and that the non-RWA spectrum can be approximated by the RWA spectrum only in the range of a sufficiently small coupling constant. Unlike the one-photon counterpart, the two-photon JC model without the RWA is found to be well defined only if $2\epsilon/\omega < 1$. As a result, the dynamics of the two-photon JC model without the RWA is significantly different from its RWA counterpart. For instance, the counter-rotating term can dramatically enhance the field squeezing effect. Since quantum optics experiments are nowadays being performed with ever-increasing field intensities, the enhanced field squeezing effect may still be observable experimentally even though the typical value of the factor ϵ/ω accessible in micromaser experiments is approximately $10^{-7} \sim 10^{-8}$ [20,21]. Furthermore, another system, namely a trapped and laser-irradiated ion, has been

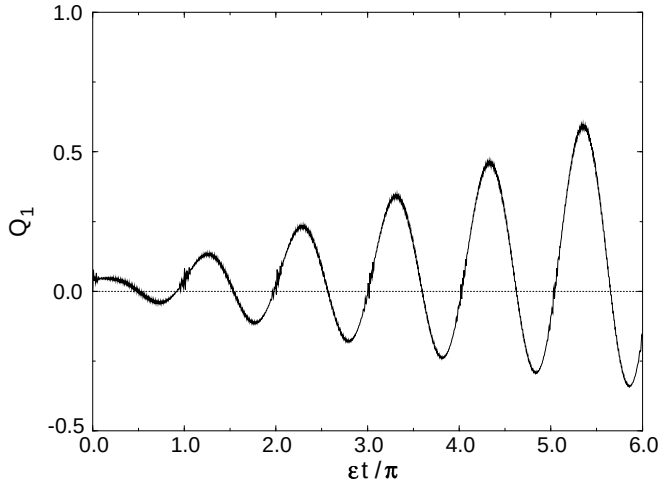

Fig. 5. The time evolution of $Q_1(t)$ for $\epsilon = 0.005$ without the RWA and at resonance.

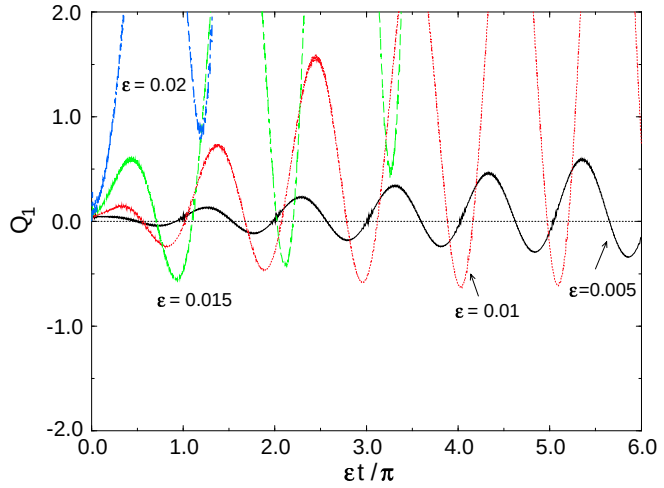

Fig. 6. The time evolution of $Q_1(t)$ for different values of the coupling parameter ϵ without the RWA and at resonance.

shown to exhibit a two-quantum JC dynamics recently [22, 23]. Here the quantized harmonic center-of-mass motion of the trapped ion plays the role of the boson mode, which is coupled via the laser to the internal (electronic) degrees of freedom. The laser-induced coupling constant is intrinsically much larger than that in quantum optics — typically ϵ/ω is about $10^{-3} \sim 10^{-4}$, and can be easily controlled by varying the intensity of the applied laser [23]. Accordingly, the effect of the counter-rotating term will be more prominent in such system. Besides, we would expect that the quantum dynamics of the two-photon JC model without the RWA is qualitatively different from that of the usual one-photon case, and we are in the process of pursuing this.

As a final remark, we would like to point out that in the context of quantum optics the RWA Hamiltonian in equation (10) can be regarded as an effective Hamiltonian for the RWA model of a three-level atom interacting with a single-field mode via two single-photon transitions, in which the Stark shift term is being neglected [24]. For atom-field interaction times which are comparable with that in the two-photon micromaser experiment, this effective Hamiltonian is able to give results similar to those of the three-level system. However, the non-RWA Hamiltonian in equation (16) may not be able to simulate the non-RWA three-level system, in which the single-photon counter-rotating terms are included, with the same degree of accuracy. This is because terms like $a^{\dagger 2}S_z$, a^2S_z, $a^\dagger aS_+$ and $a^\dagger aS_-$ are missing in the non-RWA Hamiltonian in equation (16) [25]. Nevertheless, in the context of a trapped and laser-irradiated ion the two-quantum JC model with and without the RWA is of immediate relevance to the dyanmics of the trapped ion [22,23].

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19. If $\widehat{O}$ is an operator on a Banach space $\mathcal{B}$, $|\psi\rangle$ in $\mathcal{B}$ is an analytic vector of $\widehat{O}$ if the series expansion of $\exp(\widehat{O}t)$ has a positive radius of absolute convergence; *i.e.* if $\sum_{n=0}^{\infty}(t^n/n!)\,\langle\psi|\widehat{O}^n|\psi\rangle < \infty$ for some finite $t < \infty$. See, for example, E. Nelson, Ann. Math. **70**, 572 (1959); and M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol.I: Functional Analysis* (Academic, New York, 1972), p. 201.
20. An operator $\widehat{O}$ is self-adjoint if it is Hermitian and $D(\widehat{O}) = D(\widehat{O}^\dagger)$, where $D(\widehat{O})$ and $D(\widehat{O}^\dagger)$ denote the dense domains of the operators $\widehat{O}$ and $\widehat{O}^\dagger$, respectively. Only self-adjoint

operators may be exponentiated to give unitary operators. For precise definitions and details see M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol.I: Functional Analysis* (Academic, New York, 1972), p. 255.
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25. This can be easily shown by applying the canonical transformation approach discussed in reference [24] to derive an effective Hamiltonian of the non-RWA three-level system, in which the single-photon counter-rotating terms are included.